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LETTER TO THE EDITOR

Combined variable-phase supersymmetric quantum mechanics

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Abstract. Methods of supersymmetric quantum mechanics (SUSYQM) are used to derive relations between phase functions which occur in the context of the variable-phase approach to potential scattering. It is shown that the phase-equivalent potentials obtained by the application of SUSYQM are not off-shell equivalent. A case study is presented in support of this.

In the recent past, a large number of papers appeared to combine supersymmetric quantum mechanics (SUSYQM) with the methods of formal scattering theory. The results presented are either new or older ones given a new interpretation. On a very general ground, one knows that SUSYQM is characterized by the existence of partner Hamiltonians/potentials which have identical spectra except for the missing ground state. In a remarkable paper Sukumar [1] showed that partner potentials obtained by a single SUSY transformation are phase inequivalent. However, the associated phase shifts are simply related. Interestingly, this phase-shift relationship becomes exact for the Coulomb problem. Further by making use of two successive SUSY transformations he could both eliminate bound-states and preserve the phase shift. This seminal observation has been exploited by Cooper *et al* [2], by Baye [3], by Amado *et al* [4] and by Fiedeldey *et al* [5] to study the physical processes of their personal interest.

The object of the present letter is to examine the role of SUSY transformations in the phase approach to potential scattering [6]. The phase approach often goes by the name variable-phase approach (VPA) or phase-function method (PFM). The phase approach is primarily a method for computing scattering phase shifts, and serves as a useful alternative to the Schrödinger wavefunction method. The PFM starts by assuming that the phase function $\delta(k, r)$ is the solution of the Cauchy problem

$$\delta'(k,r) = -k^{-1}v(r)\sin^2(kr+\delta(k,r) \tag{1}$$

with

$$\delta(k,0)=0. \tag{2}$$

The scattering phase is then

$$\delta(k)=\lim_{r\to\alpha}\delta(k,r). \tag{3}$$

We have written (1) for the s-wave scattering on a potential $v(r)$ at an energy $E=\frac{1}{2}k^2>0$ omitting the subscript $l=0$ on the phase function and phase shift. For the lth partial wave, however, we shall use $\delta_l(k, r)$ and $\delta_l(k)$. Here we have chosen to work with units in which $\hbar = m = 1$.

A number of generalizations of the phase approach have been considered by Calogero [6]. We propose to work with one of these and derive a combined variable-phase supersymmetric quantum mechanics (VPSSQM). In particular, we are interested in studying (i) the phase-function relationship for SUSY partners, and (ii) that for phase-equivalent potentials. As an added realism we demonstrate that the constructed phase-equivalent potentials are not off-shell equivalent.

The s-wave equation in (1) has been derived by taking $\sin kr$ and $\cos kr$ as comparison functions with respect to which the scattering phase shift is measured. Traditionally, for l-wave scattering, the comparison functions are taken as $\hat{j}_l(kr)$ and $\hat{\eta}_l(kr)$, the Riccati Bessel and Neumann functions. In contrast to this, Calogero [6] treated the centrifugal barrier as a part of the potential and wrote a generalized phase equation

$$\beta_l'(k, r) = -k^{-1}\left[\frac{l(l+1)}{2r^2} + U(r)\right]\sin^2(kr + \beta_l(k, r)) \tag{4}$$

where $\beta_l(k, 0) = 0$ and the scattering phase-shift $\delta_l(k)$ is given by

$$\delta_l(k) = \beta_l(k, a) + l\pi/2. \tag{5}$$

The significance of (4) and (5) is that the phase equation yields the phase shift due to both the centrifugal and external potential, and the scattering phase shift is obtained by subtracting the centrifugal part, $l\pi/2$, from the total phase shift $\beta_l(k, \alpha)$. The solution of (4) is subject to the additional requirement that $\beta_l'(k, 0) = -kl/(l+1)$. When the phase equation is written in the form of (4), the regular solution $\psi_l(k, r)$ of the associated Schrödinger equation and its derivative $\psi_l'(k, r)$ follow the ansatz and constraint

$$\psi_l(k, r) = \alpha_l(k, r)\sin(kr + \beta_l(k, r)) \tag{6}$$

and

$$\psi_l'(k, r) = k\alpha_l(k, r)\cos(kr + \beta_l(k, r)) \tag{7}$$

where the amplitude-function $\alpha_l(k, r)$ is related to the Jost function [7] for a potential truncated at r and is a measure of focusing or defocusing of the projectile wavefunction by the potential field [8].

Let $\psi_l(k, r)$ in (6) be the eigenfunction for a Hamiltonian H with potential $v(r)$. Then the eigenfunction for its supersymmetric partner $\tilde{H}$ with potential $\tilde{v}(r)$ can be obtained from [1]

$$\tilde{\psi}_{l+1}(k, r) = (E - E^{(0)})^{-1/2} A^- \psi_l(k, r) \tag{8}$$

where $E^{(0)}$ is the ground-state energy of H written as $E^{(0)} = -\gamma^{(0)2}/2$. The operator A^- is given by

$$A^- = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}r} + W(r)\right). \tag{9}$$

Here the 'superpotential' $W(r)$ is of the form

$$W(r) = \frac{\mathrm{d}}{\mathrm{d}r}\ln \psi_l^{(0)}(E^{(0)}, r) \tag{10}$$

with $\psi_l^{(0)}(E^{(0)}, r)$, the ground-state wavefunction of H. From (6), (7), (8) and (9) we obtain

$$\tilde{\psi}_{l+1}(E, r) = (2(E - E^{(0)}))^{-1/2} \alpha_l(k, r) \sin(kr + \beta_l(k, r) - \pi/2 + \tan^{-1}[W(r)/k]). \tag{11}$$

Further, $\tilde{\psi}_{l+1}(E, r)$ can be written in the form

$$\tilde{\psi}_{l+1}(E, r) = \tilde{\alpha}_l(k, r) \sin(kr + \tilde{\beta}_{l+1}(k, r)). \tag{12}$$

From (11) and (12) we have

$$\tilde{\alpha}_{l+1}(k, r) = (2(E - E^{(0)}))^{-1/2} \alpha_l(k, r) \tag{13}$$

and

$$\tilde{\beta}_{l+1}(k, r) = \beta_l(k, r) - \pi/2 + \tan^{-1}[W(r)/k]. \tag{14}$$

We note that (13) is equivalent to (15) of Sukumar [1]. However, our equation of interest is the phase-function relation in (14). As a useful check on (14) we see that as $r \to \alpha$, it gives the required phase-shift relationship [1].

To derive the phase-function relationship for phase-equivalent potentials we solve the partner Hamiltonian $\tilde{H}$ at the ground-state energy $E^{(0)}$ of H, i.e.

$$\tilde{H} \tilde{\psi}_l(E^{(0)}, r) = E^{(0)} \tilde{\psi}_l(E^{(0)}, r). \tag{15}$$

The wavefunction $\tilde{\psi}_l(E^{(0)}, r)$ is not square integrable, but can be chosen regular at the origin. Such a regular solution of (15) is [9]

$$\tilde{\psi}_l(E^{(0)}, r) = \frac{1}{\psi_l^{(0)}(E^{(0)}, r)} I(E^{(0)}, r) \tag{16}$$

where

$$I(E^{(0)}, r) = \int_0^r \psi_l^{(0)2}(E^{(0)}, r')\, \mathrm{d}r'. \tag{17}$$

The eigenfunction for the supersymmetric partner of $\bar{\bar{H}}$, i.e. of $\tilde{H}$ with potential $\tilde{v}(r)$ is obtained from

$$\bar{\bar{\psi}}_l(E, r) = \begin{cases} (E - E^{(0)})^{-1/2} \tilde{A}^- \tilde{\psi}_{l+1}(E, r) & (18a) \\ (E - E^{(0)})^{-1} \tilde{A}^- A^- \psi_l(E, r) & (18b) \end{cases}$$

where the operator

$$\tilde{A}^- = \frac{1}{\sqrt{2}} \left(-\frac{\mathrm{d}}{\mathrm{d}r} - W(r) + \frac{\mathrm{d}}{\mathrm{d}r} \ln I(E^{(0)}, r) \right). \tag{19}$$

From (6), (7), (18) and (19) we obtain

$$\begin{aligned} \bar{\bar{\psi}}_l(E, r) = {} & (2(E - E^{(0)}))^{-1} \alpha_l(k, r) \sin(kr + \beta_l(k, r) \\ & + \tan^{-1}(W(r)/k) - \tan^{-1}(W(r)/k - k^{-1} \frac{\mathrm{d}}{\mathrm{d}r} \ln I(E^{(0)}, r))). \end{aligned} \tag{20}$$

Further, $\bar{\bar{\psi}}_l(E, r)$ can be written in the form

$$\bar{\bar{\psi}}_l(E, r) = \bar{\bar{\alpha}}_l(k, r) \sin(kr + \bar{\bar{\beta}}_l(k, r)). \tag{21}$$

From (20) and (21) we have

$$\bar{\alpha}_l(k,r) = (2(E - E^{(0)}))^{-1}\alpha_l(k,r) \tag{22}$$

and

$$\bar{\bar{\beta}}_l(k,r) = \beta_l(k,r) + \tan^{-1}(W(r)/k) - \tan^{-1}\left(W(r)/k - k^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\ln I(E^{(0)},r)\right). \tag{23}$$

From (23) it is clear that $\bar{\bar{\beta}}_l(k,\alpha)$ and $\beta_l(k,\alpha)$ are equal. This implies that the potentials $v(r)$ and $\bar{\bar{v}}(r)$ are phase-equivalent. For finite r, $\bar{\bar{\beta}}_l(k,r)$ and $\beta_l(k,r)$ are not equal. Consequently, these potentials are wavefunction off-shell inequivalent and this results from formal elimination/addition of bound-states due to supersymmetric transformations. Note that bound states of a potential correspod to the simple poles of the off-shell two-body t matrix. Case studies based on (14) and (23) are in order and in the following we deal with this.

Our potential of interest is the three-parameter (D, r_1, d) Morse potential

$$v(r) = D\exp(2(r-r_1)/d) - 2D\exp((r-r_1)/d) \tag{24}$$

for which we recently constructed an analytical expression for the phase equivalent partner [9]. For the potential in (24) we have found

$$\bar{\beta}_1(k,r) = \beta_0(k,r) - \frac{\pi}{2} + \tan^{-1}\left[-\frac{\alpha_1 - \frac{1}{2}}{kd} + \frac{\alpha_1}{kd}\exp((r-r_1)/d)\right] \tag{25}$$

with $\alpha_1 = d\sqrt{2D}$ for the phase function relationship of $v(r)$ and $\bar{\bar{v}}(r)$. As $r \to \alpha$, (25) gives our result [9] for the scattering phase shift. The phase function relationship for $v(r)$ and $\bar{\bar{v}}(r)$ have been obtained in the form

$$\bar{\bar{\beta}}_0(k,r) = \beta_0(k,r) + \tan^{-1}\left[-\frac{\alpha_1 - \frac{1}{2}}{kd} + \frac{\alpha_1}{kd}\exp((r-r_1)/d)\right] - \tan^{-1}\left[-\frac{\alpha_1 - \frac{1}{2}}{kd} + \frac{\alpha_1}{kd}\exp((r-r_1)/d) - \frac{Z(r)}{k}\right]. \tag{26}$$

The expression for $Z(r)$ is given in our previous work [9]. Since the results in (25) and (26) refer to s-wave scattering, the βs are simply related to δs, the phase functions/ phase shifts.

In figure 1, we plot the phase functions for $v(r)$, $\bar{v}(r)$ and $\bar{\bar{v}}(r)$ as a function of r at the centre-of-mass energy $E_{\mathrm{cm}} = 100$ MeV and label them by δ, $\bar{\delta}$ and $\bar{\bar{\delta}}$, respectively. Here we have chosen to work with the potential in (24) parametrized for the nucleon–nucleon interaction in the $^3s_1$-states [10]. The phase function has at each point the meaning of the phase shift of the wavefunction for scattering by the potential at that point. The asymptotic vanishing of the potential implies that the phase function becomes asymptotically constant and this asymptotic value is just the scattering phase shift. Looking closely into our figure we see that both phase function and phase shift for $v(r)$ and $\bar{v}(r)$ are widely different, whereas, $v(r)$ and $\bar{\bar{v}}(r)$ are only off-shell inequivalent and the phase-equivalence is clearly displayed by the asymptotic values of δ and $\bar{\bar{\delta}}$. We feel that construction of on- and off-shell equivalent potentials by the method of SUSY quantum mechanics constitutes a problem of considerable interest.

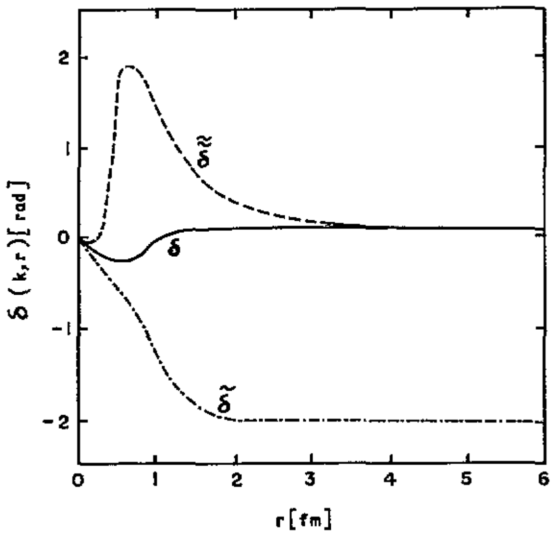

Figure 1. Phase function $\delta(k,r)$ as a function of r at $E_{cm} = 100$ MeV. The curves labelled by δ, $\tilde{\delta}$ and $\tilde{\tilde{\delta}}$ denote variation of phases induced by $v(r)$, $\tilde{v}(r)$ and $\tilde{\tilde{v}}(r)$, respectively.

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